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Alternating multivariate trigonometric functions and corresponding Fourier transforms

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Abstract

We define and study multivariate sine and cosine functions, symmetric with respect to the alternating group A_n, which is a subgroup of the permutation (symmetric) group S_n. These functions are eigenfunctions of the Laplace operator. They determine Fourier-type transforms. There exist three types of such transforms: expansions into corresponding sine-Fourier and cosine-Fourier series, integral sine-Fourier and cosine-Fourier transforms, and multivariate finite sine and cosine transforms. In all these transforms, alternating multivariate sine and cosine functions are used as a kernel.

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1. Introduction

In mathematics and mathematical physics, we very often meet functions on the Euclidean space E_n which are symmetric or antisymmetric with respect to the permutation (symmetric) group S_n. For example, such functions describe collections of identical particles. Symmetric and antisymmetric solutions appear in the theory of integrable systems. Characters of finite-dimensional irreducible representations of the group $GL(n, \mathbb{C})$ and of the group $U(n)$ are symmetric functions. One meets symmetric and antisymmetric functions in the quantum theory of many-particle systems [1–3].

Appearance of (anti)symmetric functions leads to appearance of (anti)symmetric special functions. The book [4] deals with symmetric (with respect to S_n) polynomials. Symmetric and antisymmetric multivariate exponential functions were studied in [5]. Symmetric and antisymmetric multivariate sine and cosine functions were researched in [6]. In [5, 6], symmetry and antisymmetry are considered with respect to the symmetric group S_n.

However, there exists a symmetry which is so important as the symmetry with respect to the symmetric group S_n. It is the symmetry with respect to the alternating group A_n which is an invariant subgroup of S_n of index 2 (that is, the group S_n/A_n has two elements). The alternating group A_n consists of transformations $w \in S_n$ with $\det w = 1$, that is, A_n consists of even permutations of S_n. The group A_n is a subgroup of the rotation group $SO(n)$ (note that S_n does not belong to $SO(n)$). The group A_n is simple, that is, it has no invariant subgroups.

In [7], we studied multivariate exponential functions symmetric with respect to the group A_n. We call these functions alternating multivariate exponential functions. The aim of this paper is to describe and to study multivariate sine and cosine functions symmetrized by the alternating group A_n and the corresponding Fourier-type transforms. We call these functions *alternating multivariate sine and cosine functions* and denote them by $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$, respectively, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)) \in \mathbb{R}^n$.

Alternating multivariate sine and cosine functions are connected with symmetric and antisymmetric multivariate sine and cosine functions studied in [6]. In fact, this connection is the same as the connection of the sine and cosine functions of one variable with the exponential function of one variable (see section 4).

We may consider three types of alternating multivariate sine and cosine functions: (a) functions $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ with $m = (m_1, m_2, \ldots, m_n), m_i \in \mathbb{Z}$, which determine Fourier series expansions in alternating multivariate sine and cosine functions, respectively; (b) functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i \in \mathbb{R}$, which determine integral multivariate Fourier transforms; (c) functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$, where $x = (x_1, x_2, \ldots, x_n)$ take a finite set of values; they determine multivariate finite sine and cosine transforms.

Functions (b) are symmetric with respect to elements of the alternating subgroup A_n of the permutation group S_n. Moreover, the function $\mathrm{SIN}_\lambda(x)$ is antisymmetric and the function $\mathrm{COS}_\lambda(x)$ is symmetric with respect to change of a sign of any coordinate x_i. That is, symmetries of the functions (b) are described by the extended alternating group $\tilde{A}_n = A_n \times Z_2^n$, where Z_2^n is a product of n copies of the group Z_2 of changes of a sign.

Symmetries of functions (a) are described by a wider group, since sine and cosine functions of one variable $\sin 2\pi mx, \cos 2\pi mx, m \in \mathbb{Z}$, are invariant with respect to shifts $x \to x + k, k \in \mathbb{Z}$. Symmetries of functions (a) are described by elements of the extended affine alternating group $\tilde{A}_n^{\mathrm{aff}}$ which is a product of the groups A_n, T_n and Z_2^n, where T_n consists of shifts of E_n by vectors $r = (r_1, r_2, \ldots, r_n), r_j \in \mathbb{Z}$. A fundamental domain $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$ of the group $\tilde{A}_n^{\mathrm{aff}}$ is a certain bounded set in $\mathbb{R}^n$.

The functions (a) and (b) are solutions of the Laplace equation on the Euclidean space E_n or on the corresponding fundamental domain.

Functions on the fundamental domain $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$ can be expanded into series in the functions (a). These expansions are an analogue of the usual sine and cosine Fourier series for functions of one variable. Functions (b) determine symmetrized sine and cosine Fourier integral transforms on the fundamental domain $F(\tilde{A}_n)$ of the extended alternating group $\tilde{A}_n = A_n \times Z_2^n$. This domain consists of points $x \in E_n$ such that $x_1, x_2 \geqslant x_3 \geqslant x_4 \geqslant \cdots \geqslant x_n \geqslant 0$, where $x_1, x_2 \geqslant x_3$ means that $x_1 \geqslant x_3$ and $x_2 \geqslant x_3$.

Functions (c) are used to determine symmetric (with respect to A_n) finite (that is, on a finite set) trigonometric multivariate Fourier transforms. These transforms are given on grids consisting of points of the fundamental domain $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$.

The discrete Fourier transforms, determined by (anti)symmetric multivariate sine and cosine functions, studied in [6], and by alternating multivariate sine and cosine functions, studied in this paper, have a number of practically useful properties. In particular, continuous extension of the discrete transforms smoothly interpolate digital data in any dimension. Examples show that relative to the amount of available data, these transforms provide much smoother interpolation than the conventional Fourier transforms.

Symmetric and antisymmetric multivariate trigonometric functions, studied in [6], satisfy certain boundary conditions (antisymmetric trigonometric functions vanish on the boundary of the corresponding fundamental domain and the derivative of symmetric trigonometric functions with respect to the normal to the boundary of the fundamental domain vanishes on the boundary). This means that smooth functions, which are expanded in these functions, have to satisfy these conditions, that is, not each smooth function can be expanded in (anti)symmetric multivariate trigonometric functions. Alternating multivariate sine and cosine functions satisfy no boundary conditions and any smooth function can be expanded in these trigonometric functions.

(Anti)symmetric multivariate sine and cosine functions, considered in [6], as well as alternating multivariate sine and cosine functions, studied in this paper, are closely related to symmetric and antisymmetric orbit functions defined in [8, 9] and studied in detail in [10, 11]. These orbit functions are connected with the Dynkin–Coxeter diagrams of semisimple Lie algebras of rank n. Discrete orbit function transforms, corresponding to Dynkin–Coxeter diagrams of low rank, were studied and exploited in rather useful applications (see [12–18]). Clearly, our alternating multivariate sine and cosine transforms can be applied under solution of the same problems, that is, of the problems formulated on grids or lattices. But alternating multivariate sine and cosine functions are simpler than orbit functions.

Our exposition depends on properties of the alternating group and its extensions. We also use properties of semideterminants, which are closely related to determinants and antideterminants. The determinant $\det(a_{ij})_{i,j=1}^n$ of an $n \times n$ matrix $(a_{ij})_{i,j=1}^n$ is defined as

$$\det(a_{ij})_{i,j=1}^n = \sum_{w\in S_n} (\det w) a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)}$$

where S_n is the permutation (symmetric) group of n symbols $1, 2, \ldots, n$, the set $(w(1), w(2), \ldots, w(n))$ means the set $w(1, 2, \ldots, n)$, and $\det w$ denotes a determinant of the transform w, that is, $\det w = 1$ if w is an even permutation and $\det w = -1$ otherwise. Along with the determinant, we use the antideterminant $\det^+$ of the matrix $(a_{ij})_{i,j=1}^n$ which is defined as a sum of all summands entering into the expression for a determinant, taken with the sign +,

$$\det{}^+(a_{ij})_{i,j=1}^n = \sum_{w\in S_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} = \sum_{w\in S_n} a_{w(1),1} a_{w(2),2} \cdots a_{w(n),2}.$$

For the semideterminant sdet of a matrix $(a_{ij})_{i,j=1}^n$ we have

$$\mathrm{sdet}(a_{ij})_{i,j=1}^n = \tfrac{1}{2}\left(\det(a_{ij})_{i,j=1}^n + \det{}^+(a_{ij})_{i,j=1}^n\right).$$

Clearly,

$$\mathrm{sdet}(a_{ij})_{i,j=1}^n = \sum_{w\in A_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} = \sum_{w\in A_n} a_{w(1),1} a_{w(2),2} \cdots a_{w(n),n}. \tag{1}$$

The main property of semideterminants is that they are not changed under application to rows or to columns of the corresponding matrices, a permutation $w \in A_n$. But they are not invariant under application to rows or to columns permutations $w \in S_n$ such that $\det w = -1$.

2. Alternating multivariate sine and cosine functions

In this section, we introduce a new type of functions symmetric with respect to the alternating group A_n. We call these functions the alternating multivariate sine and cosine functions. They are studied in the forthcoming sections.

An alternating multivariate sine function $\mathrm{SIN}_\lambda(x)$ of $x = (x_1, x_2, \ldots, x_n)$ is defined as the function

$$\begin{aligned}\mathrm{SIN}_\lambda(x) &\equiv \mathrm{SIN}_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x_1, x_2, \ldots, x_n) := \mathrm{sdet}(\sin 2\pi\lambda_i x_j)_{i,j=1}^n \\ &\equiv \sum_{w\in A_n} \sin 2\pi\lambda_1 x_{w(1)} \sin 2\pi\lambda_2 x_{w(2)} \cdots \sin 2\pi\lambda_n x_{w(n)} \\ &= \sum_{w\in A_n} \sin 2\pi\lambda_{w(1)} x_1 \sin 2\pi\lambda_{w(2)} x_2 \cdots \sin 2\pi\lambda_{w(n)} x_n, \end{aligned} \tag{2}$$

where $(w(1), w(2), \ldots, w(n))$ means the set $w(1, 2, \ldots, n)$, and $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$ is a set of real numbers, which determines the function $\mathrm{SIN}_\lambda(x)$.

A special case of the alternating multivariate sine functions is when λ_i are integers; in this case we write $(m_1, m_2, \ldots, m_n)$ instead of $(\lambda_1, \lambda_2, \ldots, m_n)$,

$$\mathrm{SIN}_{(m_1,m_2,\ldots,m_n)}(x) = \mathrm{sdet}(\sin 2\pi m_i x_j)_{i,j=1}^n, \qquad m_i \in \mathbb{Z}.$$

An alternating multivariate cosine function $\mathrm{COS}_\lambda(x)$ of $x = (x_1, x_2, \ldots, x_n)$ is defined as the function

$$\begin{aligned}\mathrm{COS}_\lambda(x) &\equiv \mathrm{COS}_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x_1, x_2, \ldots, x_n) := \mathrm{sdet}(\cos 2\pi\lambda_i x_j)_{i,j=1}^n \\ &\equiv \sum_{w\in A_n} (\det w) \cos 2\pi\lambda_1 x_{w(1)} \cos 2\pi\lambda_1 x_{w(2)} \cdots \cos 2\pi\lambda_1 x_{w(n)} \\ &= \sum_{w\in A_n} (\det w) \cos 2\pi\lambda_{w(1)} x_1 \cos 2\pi\lambda_{w(2)} x_2 \cdots \cos 2\pi\lambda_{w(n)} x_n. \end{aligned} \tag{3}$$

The expression (1) for the semideterminant sdet does not change under applying to rows or to columns a permutation from A_n. This means that for any permutation $w \in A_n$ we have

$$\mathrm{SIN}_{w\lambda}(x) = \mathrm{SIN}_\lambda(x), \qquad \mathrm{SIN}_\lambda(wx) = \mathrm{SIN}_\lambda(x), \tag{4}$$

$$\mathrm{COS}_{w\lambda}(x) = \mathrm{COS}_\lambda(x), \qquad \mathrm{COS}_\lambda(wx) = \mathrm{COS}_\lambda(x). \tag{5}$$

Therefore, it is enough to consider only alternating sine and cosine functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that

$$\lambda_1, \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n,$$

where $\lambda_1, \lambda_2 \geqslant \lambda_3$ means that $\lambda_1 \geqslant \lambda_3$ and $\lambda_2 \geqslant \lambda_3$. Such λ are called *semidominant*. The set of all semidominant λ is denoted by D_+^e. Below, considering alternating sine and cosine functions we assume that $\lambda \in D_+^e$.

Alternating sine and cosine functions are related to symmetric and antisymmetric multivariate sine and cosine functions $\sin_\lambda^+(x)$, $\sin_\lambda^-(x)$, $\cos_\lambda^+(x)$, $\cos_\lambda^-(x)$ studied in [6]. They are determined by the formulae

$$\sin_\lambda^+(x) = \det{}^+(\sin 2\pi\lambda_i x_j)_{i,j=1}^n, \qquad \sin_\lambda^-(x) = \det(\sin 2\pi\lambda_i x_j)_{i,j=1}^n,$$

$$\cos_\lambda^+(x) = \det{}^+(\cos 2\pi\lambda_i x_j)_{i,j=1}^n, \qquad \cos_\lambda^-(x) = \det(\cos 2\pi\lambda_i x_j)_{i,j=1}^n,$$

where $\det^+$ is the antideterminant of the corresponding matrix, and λ and x are such as in (2). A connection of the functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with symmetric and antisymmetric multivariate sine and cosine functions will be considered in section 4.

3. Extended affine alternating group and fundamental domains

In order to study symmetries of alternating multivariate sine and cosine functions, we introduce in this section the extended affine alternating group and the extended alternating group. Fundamental domains of these groups in the n-dimensional Euclidean space are derived.

We have seen (see (4) and (5)) that the functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ are symmetric with respect to the alternating group A_n. However, these functions are symmetric with respect to a wider group.

The sine and cosine functions of one variable are symmetric with respect of the operation ε of change of coordinate sign,

$$\varepsilon \sin 2\pi r y := \sin 2\pi r(-y) = -\sin 2\pi r y, \qquad \varepsilon \cos 2\pi r y = \cos 2\pi r y.$$

This symmetry is reflected in properties of the functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$. Let ε_i denote the operation of change of a sign of the coordinate x_i. One can see from the expressions (2), (3) for $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ that

$$\mathrm{SIN}_\lambda(\varepsilon_i x) = -\mathrm{SIN}_\lambda(x), \qquad \mathrm{COS}_\lambda(\varepsilon_i x) = \mathrm{COS}_\lambda(x). \tag{6}$$

We denote the group generated by changes of coordinate signs of $x = (x_1, x_2, \ldots, x_n)$ by Z_2^n, where Z_2 is the group of changes of a sign of one coordinate.

The group $\tilde{A}_n = A_n \times Z_2^n$ (a direct product of A_n and Z_2^n) is called the *extended alternating group*. It is a group of symmetries for the functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$.

We have the same symmetries under changes of signs in the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. In order to avoid these symmetries, we may assume that all coordinates $x_1, x_2, \ldots, x_n$ and all numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-negative.

The functions $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$ admit additional symmetries related to the periodicity of the sine and cosine functions $\sin 2\pi r y$ and $\cos 2\pi r y$, $r \in \mathbb{Z}$, $y \in \mathbb{R}$. These symmetries of $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ are described by the discrete group of shifts in the Euclidean space E_n by vectors

$$r \equiv r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n, \qquad r_i \in \mathbb{Z},$$

where $\mathbf{e}, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the unit vectors in directions of the corresponding coordinate axes. We denote this group by T_n. We have

$$\mathrm{SIN}_m(x + r) = \mathrm{SIN}_m(x), \qquad \mathrm{COS}_m(x + r) = \mathrm{COS}_m(x).$$

Permutations of A_n, the operations ε_i of changes of coordinate signs, and shifts of T_n generate a group which is denoted as $\tilde{A}_n^{\mathrm{aff}}$ and is called the *extended affine alternating group*. (The group generated by permutations of A_n and by shifts of T_n generate a group which is denoted as A_n^{aff} and is called the *affine alternating group*). Thus, the group $\tilde{A}_n^{\mathrm{aff}}$ is a product of its subgroups,

$$\tilde{A}_n^{\mathrm{aff}} = A_n \times Z_2^n \times T_n = \tilde{A}_n \times T_n,$$

where T_n is an invariant subgroup, that is, $wtw^{-1} \in T_n$ and $\varepsilon_i t \varepsilon_i^{-1} \in T_n$ for $w \in A_n, \varepsilon_i \in Z_2, t \in T_n$.

An open connected, simply connected set $F \subset \mathbb{R}^n$ is called a *fundamental domain* for the group $\tilde{A}_n^{\mathrm{aff}}$ (for the group $\tilde{A}_n$) if it does not contain equivalent points (that is, points x and x' such that $x' = gx$, where g is an element of $\tilde{A}_n^{\mathrm{aff}}$ or $\tilde{A}_n$, respectively) and if its closure contains at least one point from each $\tilde{A}_n^{\mathrm{aff}}$-orbit (from each $\tilde{A}_n$-orbit). Recall that an $\tilde{A}_n^{\mathrm{aff}}$-orbit of a point $x \in \mathbb{R}^n$ is the set of points wx, $w \in \tilde{A}_n^{\mathrm{aff}}$. Since $\tilde{A}_n^{\mathrm{aff}}$ contains the infinite subgroup T_n, an $\tilde{A}_n^{\mathrm{aff}}$-orbit is an infinite set of points. The group $\tilde{A}_n$ is finite and thus $\tilde{A}_n$-orbits are finite sets of points.

Since A_n consists of permutations w such that $\det w = 1$, the set D^e_{++} of all points $x = (x_1, x_2, \ldots, x_n)$ such that

$$x_1, x_2 > x_3 > \cdots > x_n > 0, \tag{7}$$

where $x_1, x_2 > x_3$ means that $x_1 > x_3$ and $x_2 > x_3$, is a fundamental domain for the group $\tilde{A}_n$ (we denote it as $F(\tilde{A}_n)$). The set of points $x = (x_1, x_2, \ldots, x_n) \in D^e_{++}$ such that

$$\tfrac{1}{2} > x_1, x_2 > x_3 > \cdots > x_n > 0 \tag{8}$$

is a fundamental domain for the extended affine alternating group $\tilde{A}_n^{\mathrm{aff}}$ (we denote it as $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$).

As we have seen, the multivariate alternating sine and cosine functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ are invariant with respect to the alternating group A_n and behave according to formula (6) under changes of coordinate signs. This means that it is sufficient to consider these functions only on the closure of the fundamental domain $F(\tilde{A}_n)$, that is, on the set D^e_+ of points x such that

$$x_1, x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n \geqslant 0.$$

Values of these functions on other points are received by using symmetries.

Symmetry of $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$ with respect to the extended affine alternating group $\tilde{A}_n^{\mathrm{aff}}$,

$$\mathrm{SIN}_m(wx + r) = \mathrm{SIN}_m(x), \qquad \mathrm{COS}_m(wx + r) = \mathrm{COS}_m(x)\; w \in A_n, \qquad r \in T_n, \tag{9}$$

$$\mathrm{SIN}_m(\varepsilon_i x) = -\mathrm{SIN}_m(x), \qquad \mathrm{COS}_m(\varepsilon_i x) = \mathrm{COS}_m(x), \qquad \varepsilon_i \in Z_2, \tag{10}$$

means that we may consider these functions on the closure of the fundamental domain $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$, that is, on the set of points

$$\tfrac{1}{2} \geqslant x_1, x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n \geqslant 0.$$

Values of these functions on other points are obtained by using relations (9) and (10).

4. Relation to symmetric and antisymmetric sine and cosine functions

The alternating multivariate sine and cosine functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ are related to symmetric and antisymmetric multivariate sine and cosine functions $\sin^+_\lambda(x)$, $\sin^-_\lambda(x)$, $\cos^+_\lambda(x)$, $\cos^-_\lambda(x)$ studied in [6].

It follows from the definitions of alternating and symmetric and antisymmetric multivariate sine and cosine functions that for λ such that $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$ we have

$$\sin^-_\lambda(x) = \mathrm{SIN}_\lambda(x) - \mathrm{SIN}_{r_{12}\lambda}(x), \qquad \sin^+_\lambda(x) = \mathrm{SIN}_\lambda(x) + \mathrm{SIN}_{r_{12}\lambda}(x),$$

$$\cos^-_\lambda(x) = \mathrm{COS}_\lambda(x) - \mathrm{COS}_{r_{12}\lambda}(x), \qquad \cos^+_\lambda(x) = \mathrm{COS}_\lambda(x) + \mathrm{COS}_{r_{12}\lambda}(x),$$

where r_{12} means the permutation of λ_1 and λ_2. It follows from here that

$$\mathrm{SIN}_\lambda(x) = \tfrac{1}{2}(\sin^+_\lambda(x) + \sin^-_\lambda(x)), \qquad \mathrm{COS}_\lambda(x) = \tfrac{1}{2}(\cos^+_\lambda(x) + \cos^-_\lambda(x)), \tag{11}$$

$$\mathrm{SIN}_{r_{12}\lambda}(x) = \tfrac{1}{2}(\sin^+_\lambda(x) - \sin^-_\lambda(x)), \qquad \mathrm{COS}_{r_{12}\lambda}(x) = \tfrac{1}{2}(\cos^+_\lambda(x) - \cos^-_\lambda(x)). \tag{12}$$

It is directly derived from these formulae that

$$\big(\sin^+_\lambda(x)\big)^2 - (\sin^-_\lambda(x))^2 = 4\mathrm{SIN}_\lambda(x)\mathrm{SIN}_{r_{12}\lambda}(x),$$

$$\left(\cos_\lambda^+(x)\right)^2 - (\cos_\lambda^-(x))^2 = 4\mathrm{COS}_\lambda(x)\mathrm{COS}_{r_{12}\lambda}(x),$$
$$\left(\sin_\lambda^+(x)\right)^2 + (\sin_\lambda^-(x))^2 = 2(\mathrm{SIN}_\lambda(x))^2 + 2(\mathrm{SIN}_{r_{12}\lambda}(x))^2,$$
$$\left(\cos_\lambda^+(x)\right)^2 + (\cos_\lambda^-(x))^2 = 2(\mathrm{COS}_\lambda(x))^2 + 2(\mathrm{COS}_{r_{12}\lambda}(x))^2.$$

If in the set $\lambda_1, \lambda_2, \ldots, \lambda_n$ there are two coinciding numbers, then due to properties of the determinant of a matrix we have $\sin_\lambda^-(x) = \cos_\lambda^-(x) = 0$. One can directly check that in this case

$$\mathrm{SIN}_\lambda(x) = \tfrac{1}{2}\sin_\lambda^+(x), \qquad \mathrm{COS}_\lambda(x) = \tfrac{1}{2}\cos_\lambda^+(x). \tag{13}$$

5. Properties

Symmetry of alternating sine and cosine functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with respect to the extended alternating group $\tilde{A}_n$ is a main property of these functions. However, they possess many other interesting properties.

Continuity and scaling symmetry. The functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ are finite sums of products of sine and cosine functions of one variable. Therefore, they are continuous functions of $x_1, x_2, \ldots, x_n$ and have continuous derivatives of all orders in $\mathbb{R}^n$. Moreover, they are real functions of $x \in \mathbb{R}^n$.

For $c \in \mathbb{R}$, let $c\lambda = (c\lambda_1, c\lambda_2, \ldots, c\lambda_n)$. Then

$$\mathrm{SIN}_{c\lambda}(x) = \mathrm{sdet}(\sin 2\pi(c\lambda_i)x_j)_{i,j=1}^n = \mathrm{sdet}(\sin 2\pi\lambda_i(cx_j))_{i,j=1}^n = \mathrm{SIN}_\lambda(cx.)$$

The equality $\mathrm{SIN}_{c\lambda}(x) = \mathrm{SIN}_\lambda(cx)$ expresses the *scaling symmetry of the functions* $\mathrm{SIN}_\lambda(x)$. Similarly, we have $\mathrm{COS}_{c\lambda}(x) = \mathrm{COS}_\lambda(cx)$.

It follows from formulae for alternating sine and cosine functions that

$$\mathrm{SIN}_\lambda(x) = \mathrm{SIN}_x(\lambda), \qquad \mathrm{COS}_\lambda(x) = \mathrm{COS}_x(\lambda).$$

Orthogonality on the fundamental domain $F\left(\tilde{A}_n^{\mathrm{aff}}\right)$. Alternating multivariate sine functions $\mathrm{SIN}_m(x)$ with $m = (m_1, m_2, \ldots, m_n) \in D_+^e, m_j \in \mathbb{Z}$, are orthogonal on the fundamental domain $F\left(\tilde{A}_n^{\mathrm{aff}}\right)$ with respect to the Euclidean measure. We have

$$2^{2n}\int_{\overline{F(\tilde{A}_n^{\mathrm{aff}})}} \mathrm{SIN}_m(x)\mathrm{SIN}_{m'}(x)\,\mathrm{d}x = |G_m|\delta_{m,m'}, \tag{14}$$

where $|G_m|$ is the order of the subgroup G_m of the alternating group A_n consisting of elements $w \in A_n$ leaving m invariant, and the closure $\overline{F\left(\tilde{A}_n^{\mathrm{aff}}\right)}$ of $F\left(\tilde{A}_n^{\mathrm{aff}}\right)$ consists of points $x = (x_1, x_2, \ldots, x_n) \in E_n$ such that

$$\tfrac{1}{2} \geqslant x_1, x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n \geqslant 0.$$

This relation follows from orthogonality of the sine functions $\sin 2\pi m_i x_j$ of one variable (entering into the definition of the function $\mathrm{SIN}_m(x)$). Indeed, we have

$$2^2\int_0^{1/2} \sin(2\pi kt)\sin(2\pi k't)\,\mathrm{d}t = \delta_{kk'},\ k, k' \in \mathbb{Z}^{>0}.$$

Let T be the set $\left[0, \tfrac{1}{2}\right]^n$. If the set $m = (m_1, m_2, \ldots, m_n)$ has no coinciding numbers, then

$$2^{2n}\int_{\mathsf{T}} \mathrm{SIN}_m(x)\mathrm{SIN}_{m'}(x)\,\mathrm{d}x = |A_n|\delta_{m,m'},$$

where $|A_n|$ is an order of the alternating group. Since we have to take $F\left(\tilde{A}_n^{\mathrm{aff}}\right)$ exactly $|A_n|$ times in order to cover the set T, the formula (14) follows for such sets $m = (m_1, m_2, \ldots, m_n)$.

If in $m = (m_1, m_2, \ldots, m_n)$ there are coinciding numbers, then in the expression (2) for $\mathrm{SIN}_m(x)$ there are coinciding terms. This leads to the multiplier $|G_m|$ in (14).

A similar orthogonality relation can be written down for the alternating multivariate cosine functions:

$$2^{2n} \int_{F(\tilde{A}_n^{\mathrm{aff}})} \mathrm{COS}_m(x) \mathrm{COS}_{m'}(x)\, \mathrm{d}x = |G_m| \delta_{m,m'}. \tag{15}$$

Solutions of the Laplace equation. The Laplace operator on the Euclidean space E_n in the Cartesian coordinates $x = (x_1, x_2, \ldots, x_n)$ has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Taking any summand in the expression for the alternating multivariate sine function, we get

$$\Delta \sin 2\pi (w\lambda)_1 x_1 \sin 2\pi (w\lambda)_2 x_2 \cdots \sin 2\pi (w\lambda)_n x_n$$
$$= -4\pi^2 \langle \lambda, \lambda \rangle \sin 2\pi (w\lambda)_1 x_1 \sin 2\pi (w\lambda)_2 x_2 \cdots \sin 2\pi (w\lambda)_n x_n,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ determines $\mathrm{SIN}_\lambda(x)$ and $\langle \lambda, \lambda \rangle = \sum_{i=1}^n \lambda^2$. The similar relation is true for summands from the expression for the alternating multivariate cosine functions. Since the action of Δ does not depend on a summand from the expression for alternating multivariate sine or cosine function, we have

$$\Delta \mathrm{SIN}_\lambda(x) = -4\pi^2 \langle \lambda, \lambda \rangle \mathrm{SIN}_\lambda(x), \qquad \Delta \mathrm{COS}_\lambda(x) = -4\pi^2 \langle \lambda, \lambda \rangle \mathrm{COS}_\lambda(x). \tag{16}$$

Symmetric and antisymmetric multivariate sine and cosine functions of [6] also satisfy these equations. Besides, they satisfy the certain boundary conditions (antisymmetric sine and cosine functions vanish on the boundary of the corresponding fundamental domain and the derivative of the symmetric sine and cosine functions with respect to the normal to the boundary of the fundamental domain vanishes on the boundary). Alternating multivariate sine and cosine functions do not satisfy these conditions.

6. Expansions in alternating sine and cosine functions on $F(\tilde{A}_n^{\mathrm{aff}})$

Alternating sine and cosine functions determine symmetric (with respect to A_n) multivariate Fourier transforms which generalize the usual sine Fourier and cosine Fourier transforms. There are three types of such transforms: (a) Fourier transforms related to the functions $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ with $m = (m_1, m_2, \ldots, m_n), m_j \in \mathbb{Z}$ (Fourier series); (b) integral Fourier transforms related to $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with $\lambda \in D_+^e$; (c) multivariate finite sine and cosine transforms.

In this section, we consider expansions in alternating sine and cosine functions $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ on the fundamental domain $F(\tilde{A}_n^{\mathrm{aff}})$. These expansions are constructed in the same way as in the case of (anti)symmetric sine and cosine functions in [6].

Let $f(x)$ be a symmetric (with respect to the extended affine alternating group $\tilde{A}_n^{\mathrm{aff}}$) continuous real function on the n-dimensional Euclidean space E_n, which has continuous derivatives. We may consider this function on the set $\mathsf{T} = \left[0, \frac{1}{2}\right]^n$ (this set is a closure of the union of the sets $wF(\tilde{A}_n^{\mathrm{aff}})$, $w \in A_n$). Then $f(x)$, as a function on T, can be expanded in sine functions

$$\sin 2\pi m_1 x_1 \cdot \sin 2\pi m_2 x_2 \cdots \sin 2\pi m_n x_n, \qquad m_i \in \mathbb{Z}^{>0}.$$

We have

$$f(x) = \sum_{m_i \in \mathbb{Z}^{>0}} c_m \sin 2\pi m_1 x_1 \cdot \sin 2\pi m_2 x_2 \cdots \sin 2\pi m_n x_n, \tag{17}$$

where $m = (m_1, m_2, \ldots, m_n)$. Let us show that $c_{wm} = c_m$, $w \in A_n$. We represent each sine function in the expression (17) in the form $\sin\alpha = (2\mathrm{i})^{-1}(\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha})$. Then

$$f(x) = \sum_{m_i \in \mathbb{Z}} c_m \,\mathrm{e}^{2\pi \mathrm{i} m_1 x_1}\, \mathrm{e}^{2\pi \mathrm{i} m_2 x_2} \cdots \mathrm{e}^{2\pi \mathrm{i} m_n x_n} = \sum_{m_i \in \mathbb{Z}} c_m \,\mathrm{e}^{2\pi \mathrm{i}\langle m, x\rangle},$$

where $\langle m, x\rangle = \sum_{i=1}^{n} m_i x_i$ and c_m with positive m_i, $i = 1, 2, \ldots, n$, are such as in (17) and each change of a sign in m leads to multiplication of c_m by (-1). Due to the property $f(wx) = f(x)$, $w \in A_n$, for any $w \in A_n$ we have

$$f(wx) = \sum_{m_i \in \mathbb{Z}} c_m \,\mathrm{e}^{2\pi \mathrm{i}\langle m, wx\rangle} = \sum_{m_i \in \mathbb{Z}} c_m \,\mathrm{e}^{2\pi \mathrm{i}\langle w^{-1} m, x\rangle}$$
$$= \sum_{m_i \in \mathbb{Z}} c_{wm} \,\mathrm{e}^{2\pi \mathrm{i}\langle m, x\rangle} = f(x) = \sum_{m_i \in \mathbb{Z}} c_m \,\mathrm{e}^{2\pi \mathrm{i}\langle m, x\rangle}.$$

The last two rows show that the coefficients c_m in (17) satisfy the conditions $c_{wm} = c_m$, $w \in A_n$.

Collections of products of sine functions of one variable at c_{wm}, $w \in A_n$, in (17) coincide with the functions $\mathrm{SIN}_m(x)$. Therefore, we obtain the expansion

$$f(x) = \sum_{m \in P_+^e} c_m \,\mathrm{sdet}(\sin 2\pi m_i x_j)_{i,j=1}^{n} \equiv \sum_{m \in P_+^e} c_m \mathrm{SIN}_m(x), \tag{18}$$

where $P_+^e := D_+^e \cap \mathbb{Z}^n$. Thus, *any symmetric (with respect to* A_n*) continuous real function* f *on* T*, which has continuous derivatives, can be expanded in antisymmetric multivariate sine functions* $\mathrm{SIN}_m(x)$, $m \in P_+^e$. Note that symmetric (with respect to A_n) real functions f on T are in fact functions on the fundamental domain $F\big(\tilde{A}_n^{\mathrm{aff}}\big)$.

By the orthogonality relation (14), the coefficients c_m in the expansion (18) are determined by the formula

$$c_m = 2^{2n} |G_m|^{-1} \int_{\tilde{F}(A_n^{\mathrm{aff}})} f(x) \mathrm{SIN}_m(x)\, \mathrm{d}x. \tag{19}$$

Moreover, the Plancherel formula

$$\sum_{m \in P_+^e} |G_m| |c_m|^2 = 2^{2n} \int_{\tilde{F}(A_n^{\mathrm{aff}})} |f(x)|^2 \,\mathrm{d}x \tag{20}$$

holds, which means that the Hilbert spaces with the appropriate scalar products are isometric.

Formula (19) is an alternating sine Fourier transform of the function $f(x)$. Formula (18) gives an inverse transform. Formulae (18) and (19) give the *alternating multivariate sine Fourier transforms* corresponding to alternating sine functions $\mathrm{SIN}_m(x)$, $m \in P_+^e$.

Let $\mathcal{L}^2\big(\overline{F\big(\tilde{A}_n^{\mathrm{aff}}\big)}\big)$ denote the Hilbert space of functions on the domain $\overline{F\big(\tilde{A}_n^{\mathrm{aff}}\big)}$ with the scalar product

$$\langle f_1, f_2\rangle = \int_{F(\tilde{A}_n^{\mathrm{aff}})} f_1(x) \overline{f_2(x)}\, \mathrm{d}x.$$

The formulae (18)–(20) show that *the set of alternating multivariate sine functions* $\mathrm{SIN}_m(x)$, $m \in P_+^e$, *forms an orthogonal basis of* $\mathcal{L}^2\big(\overline{F\big(A_n^{\mathrm{aff}}\big)}\big)$.

Analogous transforms hold for alternating cosine functions $\mathrm{COS}_m(x)$, $m \in P_+^e$. Let $f(x)$ be a symmetric (with respect to the group $\tilde{A}_n^{\mathrm{aff}}$) continuous real function on the n-dimensional

Euclidean space E_n, which has continuous derivatives. We may consider this function as a function on $F\big(\tilde{A}_n^{\rm aff}\big)$. Then we can expand this function as

$$f(x) = \sum_{m\in P_+^e} c_m \,\mathrm{sdet}(\cos 2\pi m_i x_j)_{i,j=1}^n = \sum_{m\in P_+^e} c_m \mathrm{COS}_m(x). \tag{21}$$

The coefficients c_m of this expansion are given by the formula

$$c_m = 2^{2n}|G_m|^{-1}\int_{F\left(\tilde{A}_n^{\rm aff}\right)} f(x)\mathrm{COS}_m(x)\,\mathrm{d}x. \tag{22}$$

The Plancherel formula is of the form

$$\sum_{m\in P_+^e} |G_m||c_m|^2 = 2^{2n}\int_{F\left(\tilde{A}_n^{\rm aff}\right)} |f(x)|^2\,\mathrm{d}x.$$

7. Fourier transforms on the fundamental domain $F(\tilde{A}_n)$

The expansions (18) and (21) of functions on the fundamental domain $F\big(\tilde{A}_n^{\rm aff}\big)$ are expansions in the sine and cosine functions $\mathrm{SIN}_m(x)$ and $\mathrm{COS}_m(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$. The functions $\mathrm{SIN}_\lambda(x)$ and $\mathrm{COS}_\lambda(x)$ with λ lying in the domain D_+^e (and not obligatory integral) are not invariant with respect to the corresponding affine group $\tilde{A}_n^{\rm aff}$. They are invariant only with respect to the group $\tilde{A}_n$. The closure of the fundamental domain $F(\tilde{A}_n)$ coincides with the set D_+^e consisting of the points x such that $x_1, x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n$. The functions $\mathrm{SIN}_\lambda(x)$, $\lambda \in D_+^e$, determine a Fourier-type transform on D_+^e.

We begin with the usual sine Fourier transform on $\mathbb{R}_+^n$:

$$\tilde{f}(\lambda) = \int_{\mathbb{R}_+^n} f(x)\sin 2\pi\lambda_1 x_1 \sin 2\pi\lambda_2 x_2 \cdots \sin 2\pi\lambda_n x_n\,\mathrm{d}x, \tag{23}$$

$$f(x) = 2^{2n}\int_{\mathbb{R}_+^n} \tilde{f}(\lambda)\sin 2\pi\lambda_1 x_1 \sin 2\pi\lambda_2 x_2 \cdots \sin 2\pi\lambda_n x_n\,\mathrm{d}\lambda. \tag{24}$$

Let the function $f(x)$, given on $\mathbb{R}_+^n$, be invariant with respect to the alternating group A_n, that is, $f(wx) = f(x)$, $w \in A_n$. The function $\tilde{f}(\lambda)$ is also invariant with respect to A_n:

$$\begin{aligned}\tilde{f}(w\lambda) &= \int_{\mathbb{R}_+^n} f(x)\sin 2\pi(w\lambda)_1 x_1 \cdots \sin 2\pi(w\lambda)_n x_n\,\mathrm{d}x\\ &= \int_{\mathbb{R}_+^n} f(x)\sin 2\pi\lambda_1(w^{-1}x)_1 \cdots \sin 2\pi\lambda_n(w^{-1}x)_n d(w^{-1}x)\\ &= \int_{\mathbb{R}_+^n} f(wx)\sin 2\pi\lambda_1 x_1 \cdots \sin 2\pi\lambda_n x_n\,\mathrm{d}x = \tilde{f}(\lambda).\end{aligned} \tag{25}$$

Replace λ by $w\lambda$, $w \in A_n$, on both sides of (23) and then sum up these both sides over $w \in A_n$. Due to the expression (2) for alternating sine functions $\mathrm{SIN}_\lambda(x)$, instead of (23) we obtain

$$\tilde{f}(\lambda) = |A_n|^{-1}\int_{\mathbb{R}_+^n} f(x)\mathrm{SIN}_\lambda(x)\,\mathrm{d}x \equiv \int_{D_+^e} f(x)\mathrm{SIN}_\lambda(x)\,\mathrm{d}x, \qquad \lambda \in D_+^e, \tag{26}$$

where we have taken into account that $f(x)$ is invariant with respect to A_n.

Starting from (24), we obtain the inverse formula,

$$f(x) = 2^{2n}\int_{D_+^e} \tilde{f}(\lambda)\mathrm{SIN}_\lambda(x)\,\mathrm{d}\lambda. \tag{27}$$

For the transforms (26) and (27) the Plancherel formula

$$\int_{D_+^e} |f(x)|^2 \,\mathrm{d}x = 2^{2n} \int_{D_+^e} |\tilde{f}(\lambda)|^2 \,\mathrm{d}\lambda$$

holds. The formulae (26) and (27) determine the *alternating multivariate sine Fourier transforms on the domain* $F(\tilde{A}_n)$.

The cosine functions $\mathrm{COS}_\lambda(x)$ determine similar transforms. Namely, we have

$$\tilde{f}(\lambda) = \int_{D_+^e} f(x)\mathrm{COS}_\lambda(x)\,\mathrm{d}x, \qquad \text{where} \quad f(x) = 2^{2n} \int_{D_+^e} \tilde{f}(\lambda)\mathrm{COS}_\lambda(x)\,\mathrm{d}\lambda. \tag{28}$$

The corresponding Plancherel formula holds.

8. Discrete one-dimensional cosine transforms

Discrete one-dimensional sine and cosine transforms are useful for applications. The theory of these transforms as well as their different applications and methods of work with them are given in [19] (see also [20]). In this section, we give these one-dimensional transforms in the form which will be used in the following sections.

In [19], the discrete cosine transforms are denoted as DCT-1, DCT-2, DCT-3, DCT-4. Let us expose all these transforms, conserving notations used in the literature on signal processing. They are determined by a positive integer N.

DCT-1. This transform is given by the kernel

$$\mu_r(k) = \sqrt{c_r c_k}\left(\frac{2}{N}\right)^{1/2} \cos\frac{\pi r k}{N}, \qquad \text{where} \quad k, r \in \{0, 1, 2, \ldots, N\} \tag{29}$$

where $c_k = \frac{1}{2}$ for $k = 0, N$ and $c_k = 1$ otherwise. The matrix $(\mu_r(k))_{r,k=0}^N$ is orthogonal. Therefore, the orthogonality relation for these discrete functions is given by

$$\frac{2}{N}\sum_{k=0}^{N} c_k \cos\frac{\pi r k}{N} \cos\frac{\pi r' k}{N} = c_r^{-1}\delta_{rr'}. \tag{30}$$

Thus, these functions give the expansion

$$f(k) = \sum_{r=0}^{N} a_r \cos\frac{\pi r k}{N}, \qquad \text{where} \quad a_r = \frac{2c_r}{N}\sum_{k=0}^{N} c_k f(k) \cos\frac{\pi r k}{N}. \tag{31}$$

DCT-2. This transform is given by the kernel

$$\omega_r(k) = \sqrt{c_k}\left(\frac{2}{N}\right)^{1/2} \cos\frac{\pi\left(r+\frac{1}{2}\right)k}{N}, \qquad \text{where} \quad k, r \in \{0, 1, 2, \ldots, N-1\},$$

where $c_k = 1/2$ for $k = 0$ and $c_k = 1$ otherwise.

The orthogonality relation for these discrete functions is given by

$$\frac{2}{N}\sum_{k=0}^{N-1} c_k \cos\frac{\pi\left(r+\frac{1}{2}\right)k}{N} \cos\frac{\pi\left(r'+\frac{1}{2}\right)k}{N} = \delta_{rr'}. \tag{32}$$

These functions determine the expansion

$$f(k) = \sum_{r=0}^{N-1} a_r \cos\frac{\pi\left(r+\frac{1}{2}\right)k}{N}, \qquad \text{where} \quad a_r = \frac{2}{N}\sum_{k=0}^{N-1} c_k f(k) \cos\frac{\pi\left(r+\frac{1}{2}\right)k}{N}. \tag{33}$$

DCT-3. This transform is determined by the kernel

$$\sigma_r(k) = \sqrt{c_r}\left(\frac{2}{N}\right)^{1/2}\cos\frac{\pi r\left(k+\frac{1}{2}\right)}{N},$$

where k and r run over the values $\{0, 1, 2, \ldots, N-1\}$ and where $c_r = 1/2$ for $r = 0$ and $c_r = 1$ otherwise.

The orthogonality relation for these discrete functions is given by the formula

$$\frac{2}{N}\sum_{k=0}^{N-1}\cos\frac{\pi r\left(k+\frac{1}{2}\right)}{N}\cos\frac{\pi r'\left(k+\frac{1}{2}\right)}{N} = c_r^{-1}\delta_{rr'}. \tag{34}$$

These functions give the expansion

$$f(k) = \sum_{r=0}^{N-1} a_r\cos\frac{\pi r\left(k+\frac{1}{2}\right)}{N}, \qquad \text{where} \quad a_r = \frac{2c_r}{N}\sum_{k=0}^{N-1} f(k)\cos\frac{\pi r\left(k+\frac{1}{2}\right)}{N}. \tag{35}$$

DCT-4. This transform is given by the kernel

$$\tau_r(k) = \left(\frac{2}{N}\right)^{1/2}\cos\frac{\pi\left(r+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{N},$$

where k and r run over the values $\{0, 1, 2, \ldots, N-1\}$. The orthogonality relation for these discrete functions is given by

$$\frac{2}{N}\sum_{k=0}^{N-1}\cos\frac{\pi\left(r+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{N}\cos\frac{\pi\left(r'+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{N} = \delta_{rr'}. \tag{36}$$

These functions determine the expansion

$$f(k) = \sum_{r=0}^{N-1} a_r\cos\frac{\pi\left(r+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{N}, \qquad \text{where} \quad a_r = \frac{2}{N}\sum_{k=0}^{N-1} f(k)\cos\frac{\pi\left(r+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)}{N}. \tag{37}$$

Note that there exist also four discrete sine transforms, corresponding to the above discrete cosine transforms. They are obtained from the cosine transforms by replacing in (31), (33), (35) and (37) cosines discrete functions by sine discrete functions (see [19]).

9. Alternating multivariate discrete cosine transforms

To each of the finite cosine transforms DCT-1, DCT-2, DCT-3, DCT-4 there corresponds an alternating multivariate discrete cosine transform. We denote the corresponding transforms as AMDCT-1, AMDCT-2, AMDCT-3, AMDCT-4. We fix a positive integer N and use the notation $\breve{D}_N^n$ for the subset of the set $D_N^n \equiv D_N \times D_N \times \ldots \times D_N$ (n times) with $D_N = \{0, 1, 2, \ldots, N\}$, consisting of points $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, $r_i \in \mathbb{Z}^{\geqslant 0}$, such that

$$N \geqslant r_1, r_2 \geqslant r_3 \geqslant \cdots \geqslant r_n \geqslant 0.$$

The set $\breve{F}_N^n \equiv \frac{1}{N}\breve{D}_N^n$ is a grid in the fundamental domain $F\left(\tilde{A}_n^{\text{aff}}\right)$ of the extended affine alternating group $\tilde{A}_n^{\text{aff}}$.

The set D_N^n is obtained by action by elements of the group A_n upon $\breve{D}_N^n$, that is, D_N^n coincides with the set $\left\{w\breve{D}_N^n;\, w \in A_n\right\}$. However, in $\left\{w\breve{D}_N^n;\, w \in A_n\right\}$, some points are met several times. Namely, a point $\mathbf{k}_0 \in \breve{D}_N^n$ is met $\left|A_{\mathbf{k}_0}\right|$ times in the set $\left\{w\breve{D}_N^n;\, w \in A_n\right\}$, where $\left|A_{\mathbf{k}_0}\right|$ is an order of the subgroup $A_{\mathbf{k}_0} \subset A_n$ consisting of elements $w \in A_n$ leaving $\mathbf{k}_0$ invariant.

AMDCT-1. We take the finite cosine functions (29) and make multivariate finite cosine functions by multiplying n copies of these functions:

$$\cos_{\mathbf{m}} \frac{\mathbf{s}}{N} := \cos \frac{\pi m_1 s_1}{N} \cos \frac{\pi m_2 s_2}{N} \cdots \cos \frac{\pi m_n s_n}{N}, \tag{38}$$

where $s_j, m_i \in \{0, 1, 2, \ldots, N\}$. We consider these functions for integers m_i such that $N \geqslant m_1, m_2 \geqslant m_3 \geqslant \cdots \geqslant m_n \geqslant 0$ (these n-tuples $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ are elements of $\check{D}_N^n$) and symmetrize them by means of the group A_n. As a result, we obtain a finite version of the alternating multivariate cosine function (3):

$$\begin{aligned} \mathrm{COS}^{(1)}_{\mathbf{r}}(\mathbf{k}) &:= |A_n|^{-1/2} \sum_{w \in A_n} \cos \frac{\pi r_{w(1)} k_1}{N} \cos \frac{\pi r_{w(2)} k_2}{N} \cdots \cos \frac{\pi r_{w(n)} k_n}{N} \\ &= |A_n|^{-1/2} \mathrm{sdet} \left(\cos \frac{\pi r_i k_j}{N} \right)_{i,j=1}^{n}, \end{aligned} \tag{39}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_n), k_i \in \{0, 1, 2, \ldots, N\}$, and where $A_{\mathbf{r}}$ is the subgroup of A_n consisting of elements leaving $\mathbf{r}$ invariant. (We have here expressions $\cos \frac{\pi r_i k_j}{N}$, not $\cos 2\pi r_i k_j$ as in (3).

A scalar product of functions (38) is determined by

$$\begin{aligned} \left\langle \cos_{\mathbf{m}} \frac{\mathbf{s}}{N}, \cos_{\mathbf{m}'} \frac{\mathbf{s}}{N} \right\rangle &= \prod_{i=1}^{n} \left\langle \cos \frac{\pi m_i s_i}{N}, \cos \frac{\pi m'_i s_i}{N} \right\rangle \\ &= \prod_{i=1}^{n} \sum_{s_i=0}^{N} c_{s_i} \cos \frac{\pi m_i s_i}{N} \cos \frac{\pi m'_i s_i}{N} = \left(\frac{N}{2} \right)^n c_{m_1}^{-1} \cdots c_{m_n}^{-1} \delta_{\mathbf{m},\mathbf{m}'}, \end{aligned} \tag{40}$$

where we have taken into account formula (30). Since functions $\mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s})$ are linear combinations of functions $\cos_{\mathbf{m}'} \frac{\mathbf{s}}{N}$, then a scalar product for $\mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s})$ is also defined.

Proposition 1. *For $\mathbf{m}, \mathbf{m}' \in \check{D}_N^n$, the discrete functions (39) satisfy the orthogonality relation*

$$\begin{aligned} \langle \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}), \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}) \rangle &= \sum_{\mathbf{s} \in D_N^n} c_{\mathbf{s}} \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}) \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}) \\ &= |A_n| \sum_{\mathbf{s} \in \check{D}_N^n} |A_{\mathbf{s}}|^{-1} c_{\mathbf{s}} \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}) \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}) = \left(\frac{N}{2} \right)^n c_{\mathbf{m}}^{-1} |A_{\mathbf{m}}| \delta_{\mathbf{m}\mathbf{m}'}, \end{aligned} \tag{41}$$

where $c_{\mathbf{s}} = c_{s_1} c_{s_2} \cdots c_{s_n}$ and c_{s_i} are such as in formula (29).

Proof. Due to the orthogonality relation for the cosine functions $\cos \frac{\pi r k}{N}$ (see formula (30) we have

$$\begin{aligned} \sum_{\mathbf{s} \in D_N^n} c_{\mathbf{s}} \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}) \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}) &= \frac{|A_{\mathbf{m}}|}{|A_n|} \sum_{w \in A_n} \prod_{i=1}^{n} \sum_{s_i=0}^{N} c_{s_i} \cos \frac{\pi m_{w(i)} s_i}{N} \cos \frac{\pi m'_{w(i)} s_i}{N} \\ &= |A_{\mathbf{m}}| \left(\frac{N}{2} \right)^n c_{\mathbf{m}}^{-1} \delta_{\mathbf{m}\mathbf{m}'}, \end{aligned} \tag{42}$$

where $(m_{w(1)}, m_{w(2)}, \ldots, m_{w(n)})$ is obtained from $(m_1, m_2, \ldots, m_n)$ by action by the permutation $w \in A_n$. Since functions $\mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s})$ are symmetric with respect to A_n, we have

$$\sum_{\mathbf{s} \in D_N^n} c_{\mathbf{s}} \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}) \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}) = |A_n| \sum_{\mathbf{s} \in \check{D}_N^n} |A_{\mathbf{s}}|^{-1} c_{\mathbf{s}} \mathrm{COS}^{(1)}_{\mathbf{m}}(\mathbf{s}) \mathrm{COS}^{(1)}_{\mathbf{m}'}(\mathbf{s}).$$

This proves the proposition. □

Let f be a function on $\breve{D}_N^n$ or a symmetric (with respect to A_n) function on D_N^n. Then it can be expanded in functions (39) as

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in\breve{D}_N^n} a_{\mathbf{r}} \mathrm{COS}_{\mathbf{r}}(\mathbf{k}), \tag{43}$$

where the coefficients $a_{\mathbf{r}}$ are determined by the formula

$$a_{\mathbf{r}} = \frac{c_{\mathbf{r}}|A_n|}{|A_{\mathbf{r}}|}\left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in\breve{D}_N^n} |A_{\mathbf{k}}|^{-1} c_{\mathbf{k}} f(\mathbf{k}) \mathrm{COS}_{\mathbf{r}}(\mathbf{k}). \tag{44}$$

The Plancherel formula is

$$|A_n| \sum_{\mathbf{k}\in\breve{D}_N^n} |A_{\mathbf{k}}|^{-1} c_{\mathbf{k}} |f(\mathbf{k})|^2 = \left(\frac{N}{2}\right)^n \sum_{\mathbf{r}\in\breve{D}_N^n} c_{\mathbf{r}}^{-1} |A_{\mathbf{r}}||a_{\mathbf{r}}|^2.$$

A validity of the expansions (43) and (44) follows from the relation (41).

AMDCT-2. This transform is given by the kernel

$$\mathrm{COS}_{\mathbf{r}}^{(2)}(\mathbf{k}) = |A_n|^{-1/2} \mathrm{sdet}\left(\cos\frac{\pi\left(r_i+\frac{1}{2}\right)k_j}{N}\right)_{i,j=1}^n, \qquad \mathbf{r}\in\breve{D}_{N-1}^n, \tag{45}$$

where $\breve{D}_{N-1}^n$ is the set $\breve{D}_N^n$ with N replaced by $N-1$ and $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, $k_i \in \{0, 1, 2, \ldots, N-1\}$. The orthogonality relation for these kernels is

$$\begin{aligned}\langle \mathrm{COS}_{\mathbf{r}}^{(2)}(\mathbf{k}), \mathrm{COS}_{\mathbf{r}'}^{(2)}(\mathbf{k})\rangle &= \sum_{\mathbf{k}\in\breve{D}_{N-1}^n} \frac{|A_n|}{|A_{\mathbf{k}}|} c_{\mathbf{k}} \mathrm{COS}_{\mathbf{r}}^{(2)}(\mathbf{k}) \mathrm{COS}_{\mathbf{r}'}^{(2)}(\mathbf{k}) \\ &= \left(\frac{N}{2}\right)^n |A_{\mathbf{r}}|\delta_{\mathbf{r}\mathbf{r}'},\end{aligned} \tag{46}$$

where $c_{\mathbf{k}} = c_1c_2\cdots c_n$ and c_j are such as in (32).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in\breve{D}_{N-1}^n} a_{\mathbf{r}} \mathrm{COS}_{\mathbf{r}}^{(2)}(\mathbf{k}), \tag{47}$$

where

$$a_{\mathbf{r}} = \frac{|A_n|}{|A_{\mathbf{r}}|}\left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in\breve{D}_{N-1}^n} |A_{\mathbf{k}}|^{-1} c_{\mathbf{k}} f(\mathbf{k}) \mathrm{COS}_{\mathbf{r}}^{(2)}(\mathbf{k}).$$

The corresponding Plancherel formula holds.

AMDCT-3. This transform is given by the kernel

$$\mathrm{COS}_{\mathbf{r}}^{(3)}(\mathbf{k}) = |A_n|^{-1/2} \mathrm{sdet}\left(\cos\frac{\pi r_i\left(k_j+\frac{1}{2}\right)}{N}\right)_{i,j=1}^n, \tag{48}$$

where $\mathbf{r}\in\breve{D}_{N-1}^n$. The orthogonality relation for these kernels is

$$\begin{aligned}\langle \mathrm{COS}_{\mathbf{r}}^{(3)}(\mathbf{k}), \mathrm{COS}_{\mathbf{r}'}^{(3)}(\mathbf{k})\rangle &= \sum_{\mathbf{k}\in\breve{D}_{N-1}^n} \frac{|A_n|}{|A_{\mathbf{k}}|} \mathrm{COS}_{\mathbf{r}}^{(3)}(\mathbf{k}) \mathrm{COS}_{\mathbf{r}'}^{(3)}(\mathbf{k}) \\ &= c_{\mathbf{r}}^{-1}\left(\frac{N}{2}\right)^n |A_{\mathbf{r}}|\delta_{\mathbf{r}\mathbf{r}'},\end{aligned} \tag{49}$$

where $c_{\mathbf{r}} = c_1c_2\cdots c_n$ and c_i are such as in formula (34).

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in \check{D}^n_{N-1}} a_{\mathbf{r}} \mathrm{COS}^{(3)}_{\mathbf{r}}(\mathbf{k}), \tag{50}$$

where

$$a_{\mathbf{r}} = \frac{c_{\mathbf{r}}|A_n|}{|A_{\mathbf{r}}|}\left(\frac{2}{N}\right)^n \sum_{\mathbf{k}\in \check{D}^n_{N-1}} |A_{\mathbf{k}}|^{-1} f(\mathbf{k})\mathrm{COS}^{(3)}_{\mathbf{r}}(\mathbf{k}).$$

The corresponding Plancherel formula holds.

AMDCT-4. This transform is given by the kernel

$$\mathrm{COS}^{(4)}_{\mathbf{r}}(\mathbf{k}) = |A_n|^{-1/2}\mathrm{sdet}\left(\cos\frac{\pi\left(r_i+\frac{1}{2}\right)\left(k_j+\frac{1}{2}\right)}{N}\right)^n_{i,j=1}, \tag{51}$$

where $\mathbf{r}\in \check{D}^n_{N-1}$. The orthogonality relation for these kernels is

$$\langle \mathrm{COS}^{(4)}_{\mathbf{r}}(\mathbf{k}), \mathrm{COS}^{(4)}_{\mathbf{r}'}(\mathbf{k})\rangle = \sum_{\mathbf{k}\in \check{D}^n_{N-1}} \frac{|A_n|}{|A_{\mathbf{k}}|}\mathrm{COS}^{(4)}_{\mathbf{r}}(\mathbf{k})\mathrm{COS}^{(4)}_{\mathbf{r}'}(\mathbf{k})$$

$$= \left(\frac{N}{2}\right)^n |A_{\mathbf{r}}|\delta_{\mathbf{r}\mathbf{r}'}. \tag{52}$$

This transform is given by the formula

$$f(\mathbf{k}) = \sum_{\mathbf{r}\in \check{D}^n_{N-1}} a_{\mathbf{r}} \mathrm{COS}^{(4)}_{\mathbf{r}}(\mathbf{k}), \tag{53}$$

where

$$a_{\mathbf{r}} = \left(\frac{2}{N}\right)^n \frac{|A_n|}{|A_{\mathbf{r}}|} \sum_{\mathbf{k}\in \check{D}^n_{N-1}} |A_{\mathbf{k}}|^{-1} f(\mathbf{k})\mathrm{COS}^{(4)}_{\mathbf{r}}(\mathbf{k}).$$

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